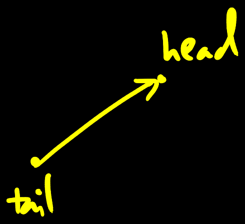
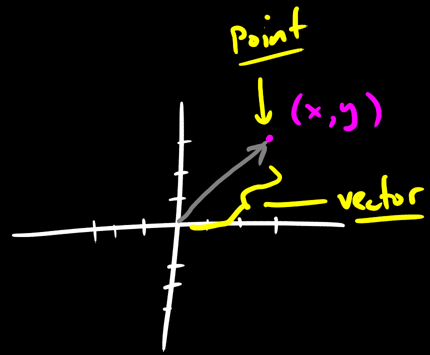
Last Time*: Matrix operations, reps of lin systems w/ mats.

Geometry

Case study: $\mathbb{R}^2$





Points: pairs (in $\mathbb{R}^2$) of real numbers.

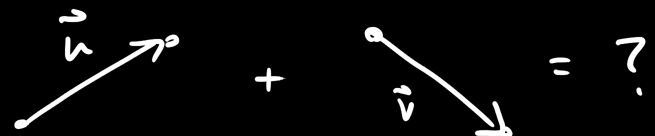
vector: directed line segment connecting two points.

↳ can be represented as a pair (in $\mathbb{R}^2$)

Vector operations: matrix operations on vectors (for the most part).

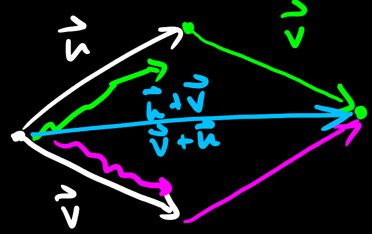
Ex: the sum of vectors $\vec{u}$ and $\vec{v}$ is the matrix sum.

Geometrically:



If $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$, then

$\vec{u} + \vec{v} = (x_1 + x_2,\ y_1 + y_2)$

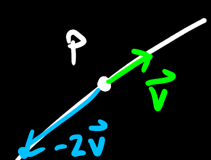


NB: These vectors live in $\mathbb{R}^2$, but in general, we'll work in $\mathbb{R}^n = \{$vectors with n components$\}$.

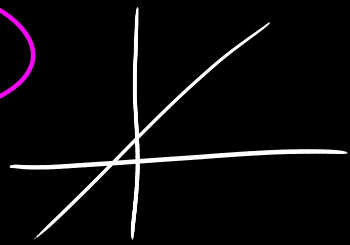
Lines: Algebraically, lines can be represented via:

Parameterization: $\{\vec{p} + t\vec{v} : t \in \mathbb{R}\}$

↳ works in $\mathbb{R}^n$

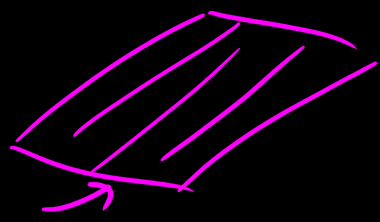


Equation (in $\mathbb{R}^2$): $\boxed{ax + by = c}$



Remark: In higher dimensions, 1 linear equation doesn't describe a line ☹

in $\mathbb{R}^3$: $ax + by + cz = d$ yields a plane



$\rightsquigarrow$ the plane parameterizes like so: $(a \neq 0)$

$$\begin{cases} x = \frac{1}{a}(d - bs - ct) \\ y = s \\ z = t \end{cases} \rightsquigarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d/a \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -b/a \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -c/a \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = d \right\}$$

$$= \left\{ \begin{bmatrix} d/a \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -b/a \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -c/a \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

↑ parameterization of our plane...

NB: 2 free variables $\rightsquigarrow$ dimension 2 $\rightsquigarrow$ 2-flat

i.e. planes are 2-flats.

Defn: A <u>k-flat</u> (in $\mathbb{R}^n$) is a k-dimensional version of a line. I.e. a set of vectors which can be expressed as:

$$\{ \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k : t_1, t_2, \ldots, t_k \in \mathbb{R}^k \}$$

For some collection of (linearly independent) vectors $v_1, v_2, \ldots, v_k$

NB: Specially named flats in $\mathbb{R}^n$

points: 0-flats
lines: 1-flats
planes: 2-flats
hyperplanes: $(n-1)$-flats

Lem: The solution set of a linear system is always a k-flat for some k.

Point: Linear systems have some rich associated geometry.

Geometry and Vector Operations

Defn: The *length* of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ is
$$|\vec{v}| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Lem: For all $\vec{v} \in \mathbb{R}^n$, $|\vec{v}| \geq 0$. Furthermore, $|\vec{v}| = 0$ precisely when $\vec{v} = \vec{0}$.

Reason: Sums of nonnegative numbers are nonnegative
squares of any (real) numbers are nonnegative
(so the square root of a sum of squares is well-defined).
principal square roots are nonnegative.
If $\sum_{i=1}^{n} v_i^2 = 0$, necessarily each $v_i = 0$.

Defn: The *dot product* (i.e. *inner product*) of vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is defined by
$$\vec{u} \cdot \vec{v} = (u_1, u_2, \ldots, u_n) \cdot (v_1, \ldots, v_n) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Lem: For all $\vec{v} \in \mathbb{R}^n$, $|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}}$. (i.e. $\vec{v}\cdot\vec{v} = |\vec{v}|^2$).

pf: Let $\vec{v} = (v_1, v_2, \cdots, v_n)$ be arbitrary. On one hand,

$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. On the other hand,

$$\sqrt{\vec{v}\cdot\vec{v}} = \sqrt{(v_1, v_2, \cdots, v_n)\cdot(v_1, v_2, \cdots, v_n)} = \sqrt{v_1 \cdot v_1 + v_2 v_2 + \cdots + v_n v_n}$$

$$= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2},$$ so $|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}}$ as desired $\boxtimes$

Ex: Let $\vec{u} = (3, 0, -1, 5)$, $\vec{v} = (-2, -3, 6, 1)$

$$\vec{u}\cdot\vec{v} = (3, 0, -1, 5)\cdot(-2, -3, 6, 1)$$

$$= 3\cdot -2 + 0 \cdot -3 + -1\cdot 6 + 5\cdot 1$$

$$= -6 + 0 - 6 + 5 = -7$$ $\boxtimes$

NB: The dot product can be thought of as a function

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

Prop (Properties of Dot Product): Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$.

① $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

pf: $$(u_1, u_2, \cdots, u_n)\cdot(v_1, v_2, \cdots, v_n) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$
$$= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$
$$= (v_1, v_2, \cdots, v_n)\cdot(u_1, u_2, \cdots, u_n).$$ $\boxtimes$

② $\vec{u}\cdot(\vec{v} + \vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ ←

pf: $$(u_1, u_2, \cdots, u_n)\cdot((v_1, v_2, \cdots, v_n) + (w_1, w_2, \cdots, w_n))$$
$$= (u_1, u_2, \cdots, u_n)\cdot(v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)$$
$$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n)$$
$$= (u_1 v_1 + u_1 w_1) + (u_2 v_2 + u_2 w_2) + \cdots + (u_n v_n + u_n w_n)$$
$$= (u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) + (u_1 w_1 + u_2 w_2 + \cdots + u_n w_n)$$
$$= (u_1, u_2, \cdots, u_n)\cdot(v_1, v_2, \cdots, v_n) + (u_1, u_2, \cdots, u_n)\cdot(w_1, w_2, \cdots, w_n)$$ $\boxtimes$

③ $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$

pf: $(c(u_1, u_2, \cdots, u_n))\cdot(v_1, v_2, \cdots, v_n)$

$= (cu_1, cu_2, \cdots, cu_n)\cdot(v_1, v_2, \cdots, v_n)$

$= (cu_1)v_1 + (cu_2)v_2 + \cdots + (cu_n)v_n$

$= c(u_1v_1) + c(u_2v_2) + \cdots + c(u_nv_n)$

$= c(u_1v_1 + u_2v_2 + \cdots + u_nv_n)$

$= c((u_1, u_2, \cdots, u_n)\cdot(v_1, v_2, \cdots, v_n))$.

So $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$. Moreover

$\vec{u}\cdot(c\vec{v}) = (c\vec{v})\cdot\vec{u} = c(\vec{v}\cdot\vec{u}) = c(\vec{u}\cdot\vec{v})$ $\boxtimes$

④ $\vec{0}\cdot\vec{v} = 0$

pf: $(0, 0, \cdots, 0)\cdot(v_1, v_2, \cdots, v_n)$

$= 0v_1 + 0v_2 + \cdots + 0v_n$

$= 0(v_1 + v_2 + \cdots + v_n)$

$= 0$ $\boxtimes$

<u>Next time</u>: Tie Geometry of dot product to the algebraic properties we just proved.